LANL xxx E-print archive No. gr-qc/9802058

arXiv:gr-qc/9802058v1 22 Feb 1998

On metric-connection compatibility and the signature change of space-time

Bozhidar Z. Iliev [*] [†]

Short title: **Metric-connection consistency and signature change**

Basic ideas:	→ April 13, 1997
Began:	→ April 15, 1997
Ended:	→ April 20, 1997
Revised:	→ September 1997, January 1998
Last update:	→ February 10, 1998
Produced:	→ $\boxed{\text{February 5, 2008}}$

LANL xxx archive server E-print No.: gr-qc/9802058

BO/HO ® TM

Subj. Class.:
General relativity; Differential geometry

1991 MSC numbers:	***1996 PACS numbers:***
53C07, 53B05, 53C80, 83D05	*02.40.Ky, 02.40.Ma, 04.50.+h*

Key-Words:
Signature change; Signature of space-time;
Metric-connection compatibility; Metric-compatible connections

[*]Department Mathematical Modeling, Institute for Nuclear Research and Nuclear Energy, Bulgarian Academy of Sciences, Boul. Tzarigradsko chaussée 72, 1784 Sofia, Bulgaria

[†]E-mail address: bozho@inrne.acad.bg

Abstract

We discuss and investigate the problem of existence of metric-compatible linear connections for a given space-time metric which is, generally, assumed to be semi-pseudo-Riemannian. We prove that under sufficiently general conditions such connections exist iff the rank and signature of the metric are constant. On this base we analyze possible changes of the space-time signature.

1 Introduction

It seems that for the first time in the works [1,2] (see also [3,4]) were discussed space-time models with a possible change of the signature of the space-time metric, called also space-time signature or simply signature. At the beginning of the nineties there appeared more often works on this subject [5–13]. Some of them study the generic mathematical structure of the space-time(s) with changing signature [6, 9, 14–18], while others deal with specific such models [10, 16, 19–23]. There are also articles investigating possible physical phenomena that can happen if the signature changes [10–12, 23–25].

The main mathematical result of this work is a necessary and sufficient condition for when a given semi-pseudo-Riemannian metric admits a compatible (metric-compatible) with it linear connection. Freely speaking, we can say that metric-compatible linear connections exist iff the rank and signature of the metric are constant. (Notice, we consider nondegenerate as well as degenerate metrics.) On this rigorous base, analyzing the conditions responsible for the signature constancy, we make conclusions for when the space-time signature can change.

Above we mentioned in parentheses that the space-time metric can be degenerate. This requires some explanations since almost everywhere a metric is defined as a non-degenerate Hermitian (resp. symmetric) quadratic form in the complex (resp. real) case. For this purpose we shall fix first some concepts. Following [26, pp. 66, 74], we call g a *pseudo-Riemannian* (resp. *pseudo-Hermitian*) *metric* in a real (resp. complex) vector bundle ξ if it is symmetric (resp. Hermitian) bilinear (resp. linear in the first and antilinear in the second argument) nondegenerate quadratic form on the fibres of ξ. The pair (ξ, g) is called pseudo-Riemannian (resp. pseudo-Hermitian) vector bundle. If g is positively defined, it is called *proper* or *Riemannian*; otherwise it is called *indefinite*. The same terminology is transferred on differentiable manifolds [27, p. 273] for which ξ is replaced with the corresponding bundles tangent to them, e.g. (M, g) is a pseudo-Riemannian manifold if g is a 2-covariant symmetric nondegenerate tensor field on M. If in the above definitions is omitted the nondegeneracy condition, to the corresponding concepts is added the prefix '*semi-*' [28] (see also the references in [28, remark 8 to chapter VI (p. 508)]); for instance, a *semi-pseudo-Riemannian metric* on a manifold M is a 2-covariant symmetric tensor field on M (see also [29, e.g., the articlle "semi-pseudo-Riemannian space"]).

Remark 1.1. A different, but analogous, terminology is used in the theory of linear partial differential equations of second order [30–36]. The type of the equation $\sum_{i,j=1}^{n} A_{ij}(x)\partial^2 u/\partial x^i \partial x^j + F(x, u, \partial u/\partial x^1, \dots, \partial u/\partial x^n) = 0$, $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ is determined by the eigenvalues of the matrix $A(x) := [A_{ij}(x)]$ [37, chapter 8, § 2] which plays the role of a metric. This equation is of type (p, q, r) at $x \in \mathbb{R}^n$ if at x the matrix $A(x)$ has p positive, q negative, and r zero eigenvalues.* The

*This classification can be extended also on non-linear partial differential equations of

type is called elliptic, parabolic, or hyperbolic at a given point if it has respectively the form $(m,0,0)$ or $(0,m,0)$, $(m-1,0,1)$ or $(0,m-1,1)$, $(m-1,1,0)$ or $(1,m-1,0)$ at that point. Using this terminology, the (proper) Riemannian and Lorentzian metrics [†] can be called respectively (globally) elliptic and (globally) hyperbolic metrics; there are no 'parabolic metrics' between the pseudo-Riemannian ones. The most widely investigated case is when equation's type is constant in the region where it is considered, e.g. in the whole space; in our analogy the pseudo-Riemannian metrics with constant signature correspond to (part of) such equations. If the equation's type changes from point to point, it is said to be of *mixed type* [30,31,34,38,39].[‡] The 'metrical' analog of these equations are pseudo-Riemannian metrics with changing (from point to point) signature, therefore they can be called *mixed (pseudo-)Riemannian* metrics. For such metrics, because of their non-degeneracy, always exist points at which they are not continuous (or smooth). However, the equations of type (p,q,r) with $r \geq 1$ do not have analogs between pseudo-Riemannian metrics as the latter are, by definition, nondegenerate. In this context, it is not difficult to see that there is a full one-to-one correspondence between the classification by type of the (linear) partial differential equations of second order and the class of semi-pseudo-Riemannian metrics. For instance, now there are parabolic metrics, defined as semi-pseudo-Riemannian ones with defect 1, i.e. with exactly 1 vanishing eigenvalue; also one can freely investigate continuous (or smooth) *mixed semi-pseudo-Riemannian* metrics, i.e. ones with changing signature and points at which they are degenerate, as they are always somewhere degenerate, etc. Ending this long remark, we want to emphasize on the fact that the revealed analogy between semi-pseudo-Riemannian metrics and the (classification of the) (linear) partial differential equation of second order is another argument for the investigation of degenerate metrics.

For the general considerations of problems concerning signature changes and metric-connection compatibility one should work with semi-pseudo-Riemannian (resp. semi-pseudo-Hermitian) metrics instead of conventional pseudo-Riemannian (resp. pseudo-Hermitian) ones in the real (resp. complex) case. This allows to be retained the concept of a global metric in some natural cases in which the conventional definition breaks down. For instance, let a manifold M be divided into several (not less then two) regions $U_i,\ i = 1, 2, \ldots$ Let on each U_i be given a pseudo-Riemannian smooth metric g_i and there to exists a symmetric tensor field g on M whose restriction on U_i is exactly g_i, $g|U_i = g_i$. It is natural to call g a (global) 'pseudo-Riemannian metric' on M. But is this 'definition' correct in the conventional sense? Yes, if the signatures of all g_i are equal and g is smooth. But if at least two (local) metrics, say g_i and g_j, have different signatures, then for g must exist regions (possibly with dimension less then $\dim M$) on which it is degenerate or/and discontinuous. In particular, if g is continuously defined

second order [37].

[†]An indefinite pseudo-Riemannian metric with exactly one positive or negative eigenvalue is called Lorentzian.

[‡]A classical example of such an equation is the Tricomi equation $\partial^2 u/\partial^2 y + y \partial^2 u/\partial^2 x = 0$.

on M, then it must be degenerate somewhere and, consequently, g is not a pseudo-Riemannian metric. Looking over the literature, we see that the last situation is not an exceptional one, on the contrary, it is the case usually considered [5, 10, 11, 20, 24]. Moreover, often g is called (intuitively) a metric in such cases nevertheless that it is a degenerate tensor field! The above considerations force us to use semi-pseudo-Riemannian metrics instead of pseudo-Riemannian ones. So, in this paper we shall omit the nondegeneracy condition in the metric's definition. Hence, the metrics investigated here can be non-degenerate as well as degenerate ones. In this way we achieve a uniform description of problems which otherwise have to be investigated separately.

In Sect. 2 we present some preliminary mathematical material on which our investigation rests. In Sect. 3 the problem for when a given metric admits a metric-compatible linear connection is rigorously analyzed. Here results concerning the more general problem on metric-compatible linear transports along paths are given too. In Sect. 4 we make conclusions on the possible changes of the space-time signature. Sect. 5 closes the paper with some concluding remarks.

2 Linear transports along paths and their compatibility with a fibre metric

In this section we recall some facts concerning linear transports along paths in vector bundles [40] and briefly review the problem of their compatibility (consistency) with a metric on the bundle [41].

Let (E, π, M) be a complex vector fibre bundle with base M, total space E, and projection $\pi : E \to M$. The fibres $E_x := \pi^{-1}(x) \subset E$, $x \in M$, are isomorphic vector spaces, i.e. there exists a vector space $\mathcal{E}$ and linear isomorphisms l_x, $x \in M$ such that $l_x : E_x \to \mathcal{E}$. We do not make any assumptions on the dimensionality of (E, π, M), i.e. $\mathcal{E}$ can be finite as well as infinite dimensional.

By J and $\gamma : J \to M$ we denote a real interval and a path in M, respectively.

A $\mathbb{C}$*-linear transport (L-transport) along paths* in (E, π, M) is a map $L : \gamma \mapsto L^\gamma$, where $L^\gamma : (s,t) \mapsto L^\gamma_{s\to t}$, $s, t \in J$ is the (L-)transport along γ, and $L^\gamma_{s\to t} : \pi^{-1}(\gamma(s)) \to \pi^{-1}(\gamma(t))$, called (L-)transport along γ from s to t, satisfies the equalities

$$L^\gamma_{t\to r} \circ L^\gamma_{s\to t} = L^\gamma_{s\to r}, \quad r, s, t \in J, \tag{2.1}$$

$$L^\gamma_{s\to s} = id_{\pi^{-1}(\gamma(s))}, \quad s \in J, \tag{2.2}$$

$$L^\gamma_{s\to t}(\lambda u + \mu v) = \lambda L^\gamma_{s\to t} u + \mu L^\gamma_{s\to t} v, \quad \mu, \lambda \in \mathbb{C}, \quad u, v \in \pi^{-1}(\gamma(s)). \tag{2.3}$$

Here id_N denotes the identity map of a set N. The general form of $L^{\gamma}_{s\to t}$ is

$$L^{\gamma}_{s\to t} = (F^{\gamma}_t)^{-1} \circ F^{\gamma}_s, \quad s,t \in J \tag{2.4}$$

with $F^{\gamma}_s : \pi^{-1}(\gamma(s)) \to Q,\ s \in J$, being one-to-one linear maps onto one and the same (complex) vector space Q.

Equations (2.1) and (2.2) imply

$$(L^{\gamma}_{s\to t})^{-1} = L^{\gamma}_{t\to s}. \tag{2.5}$$

According to [42, theorem 3.1] the set of (linear) transports which are diffeomorphisms and satisfy the locality and reparametrization conditions[§] are in one-to-one correspondence with the (axiomatically defined) (linear) parallel transports (along curves). So, the conventional parallel transport along γ from $\gamma(s)$ to $\gamma(t)$, assigned to a linear connection, is a standard realization of the general (linear) transport $L^{\gamma}_{s\to t}$.

Let g be a semi-pseudo-Hermitian fibre metric on (E, π, M). This means that $g : x \mapsto g_x$ with $g_x : E_x \times E_x \to \mathbb{C}$, $x \in M$, being Hermitian forms [28, 43], i.e. g_x are $\mathbb{C}$-linear in the first argument and $\mathbb{C}$-antilinear in the second argument Hermitian maps.

Definition 2.1 (cf. [41, definition 4.1]). *A fibre semi-pseudo-Hermitian metric g and a linear transport L are called compatible (resp. along γ) if L preserves the scalar product defined by g, i.e.*

$$g_{\gamma(s)} = g_{\gamma(t)} \circ (L^{\gamma}_{s\to t} \times L^{\gamma}_{s\to t}), \quad s,t \in J \tag{2.6}$$

for all (resp. the given) γ.

In [41] different results concerning the compatibility of linear transports along paths and fibre metrics can be found. They are prove in [41] only in the finite dimensional case, i.e. for $\dim \mathcal{E} < \infty$. However, some of them remain valid also in the infinite dimensional case. For instance, proposition 4.3 of [41] is easily shown to be insensitive to the bundle's dimensionality, i.e. for any ($\mathbb{C}$-)linear transport along paths there exist consistent with it Hermitian fibre metrics along any fixed path. The general form of these metrics is given by [41, equation (4.8)]. The global version, viz. along arbitrary paths, of this statement is not always true. In the finite dimensional case it is expressed by [41, proposition 4.6], which *mutatis mutandis* holds and for an infinite dimension.

Below we are going to use the following result (cf. [41, proposition 4.4]).

[§]I.e. respectively $L^{\gamma}_{s\to t} \in \mathrm{Diff}(\pi^{-1}(\gamma(s)), \pi^{-1}(\gamma(t)))$, $L^{\gamma|J'}_{s\to t} = L^{\gamma}_{s\to t}$ for $s,t \in J'$, with J' being a subinterval of J, and $L^{\gamma\circ\tau}_{s\to t} = L^{\gamma}_{\tau(s)\to\tau(t)}$, $s,t \in J''$ with τ being a 1:1 map of an $\mathbb{R}$-interval J'' onto J.

Proposition 2.1. *Let (E, π, B) be a finite dimensional complex vector bundle endowed with a pseudo-Hermitian fibre metric g. A necessary and sufficient condition for the existence of a compatible (resp. along a fixed path) with g ($\mathbb{C}$-)linear transport along paths (resp. along the given path) is the independence of the signature of g of the point of B (resp. of the fixed path) at which it is calculated.*

We want to make several comments on this result which is a simple reformulation of [41, proposition 4.4].

Proposition 2.1 holds also for pseudo-Riemannian metrics (see [41, proposition 2.4]) as they are evident special case of the pseudo-Hermitian ones [44].

The validity of proposition 2.1 is limited to finite dimensional vector bundles and cannot be generalized to infinite dimensional ones. There are two main reasons for this. On one hand, in its formulation is involved the notion of signature of Hermitian forms [45, section 2.12], which is generically a finite dimensional concept since it is (usually - see below) defined as the difference between the number of positive and number of negative eigenvalues of a form, i.e. between its positive and negative (inertia) indexes [46, p. 334]. On the other hand, the proof of the discussed proposition (see [41]) uses *essentially* the ((Jacobi-)Sylvester) law of inertia for Hermitian (or symmetric, in the real case) quadratic forms (see, e.g., [45, section 2.12] and [46, p. 297 and p. 334]). The law of inertia can be generalized for (degenerate of not) Hermitian (resp. symmetric) forms over ordered fields. However this is possible only for finite dimensional vector spaces [47, chapter 12, § 90].

Since the law of inertia has a form valid also for degenerate Hermitian forms [45, section 2.12], the proofs of propositions 2.4 and 4.4 of [41] can be mended *mutatis mutandis* by its help in such a way that they remain true for degenerate metrics too. This leads us to the following generalization of proposition 2.1.

Proposition 2.2. *Let (E, π, B) be a finite dimensional complex vector bundle endowed with a semi-pseudo-Hermitian fibre metric g. There exists a compatible (resp. along a fixed path) with g ($\mathbb{C}$-)linear transport along paths (resp. along the given path) if and only if the rank and signature of g are independent of the point of B (resp. of the fixed path) at which they are calculated.*

Ending this section, we pay attention to the definition of metric's signature. At $x \in B$ a metric g is represented by Hermitian form $g_x : E_x \times E_x \to \mathbb{C}$. Let $p(x)$ and $q(x)$ be its, respectively, positive and negative (inertia) indexes, which are equal to (or can be defined as) the number of positive and negative, respectively, eigenvalues of g_x, i.e. of the matrix representing g_x in some local bases. Mathematically the signature of g at x is defined as the pair $(p(x), q(x))$ or, more often, as the number $s(x) = p(x) - p(x)$ [45]. The last

definition will be used in this work.[¶] In the physical literature signature of a nondegenerate metric is called the put in parentheses sequence $(++\cdots--)$ of $p(x)$ plus signs and $q(x)$ minus signs, corresponding to the eigenvalues of the metric, in the order in which they appear in the (standard) diagonal form of the matrix representing g_x in a some local basis. This definition of the signature, that can be called physical, will not be used here.

3 Which metrics admit metric-compatible connections?

Most of the modern nonquantum gravity theories [49, 50] are constructed by means of two basic geometrical structures over the space-time (real) manifold M, viz. a pseudo-Riemannian metric g and a linear connection (covariant derivative) ∇ which is a specific derivation of the tensor algebra over M [43]. So, the metric is supposed to be a nondegenerate, symmetric, and two times contravariant tensor field [43,44], i.e. a nondegenerate section of the symmetric tensor bundle of type $(0,2)$ over M. These structures are called *compatible* (or consistent) on M if g is of class C^1 and

$$\nabla_X(g) = 0 \tag{3.1}$$

for any vector (field) X. In this case the connection ∇ is called *metric-compatible* (with the given metric on M). Examples of gravitational theories based on a metric-compatible connections are general relativity, Einstein-Cartan (U_4) theory, and Einstein teleparallelism theory [49].

Given a (C^1) pseudo-Riemannian metric g, it is known [43, chapter IV, § 2] that there always exists a metric-compatible with it linear connection (which is unique in the torsionless case). Here a natural question arises: what properties of g are responsible for this existence? Looking over the proof(s) of the existence of metric-compatible connection(s), one finds, at first sight, that they essentially use that the metric is nondegenerate and of class of smoothness not less than C^1. So, if one of these conditions breaks, one can expect a nonexistence of metric-compatible connection(s). However, as we shall see below, these are not the primary causes for such nonexistence. To examine this problem in details, we shall reformulate (3.1) in terms of parallel transports (translations).

Let $\gamma\colon [0,1] \to M$ be a C^1 path in M and $g\colon x \mapsto g_x$, $x \in M$, with $g_x\colon T_x(M) \times T_x(M) \to \mathbb{R}$. Here $T_x(M)$ is the space tangent to M at x. The linear connection ∇ can equivalently be described by the concept of parallel transport τ along paths (see [51, section 5.2] and [43]).[‖]

[¶]The law of inertia states that the numbers $p(x)$, $q(x)$, $s(x)$, and the rank $r(x)$ of g_x are invariants that are independent of the local bases by means of which they are determined [45, 48].

[‖]If ∇ is given, one can define $\tau\colon \gamma \mapsto \tau^\gamma\colon T_{\gamma(0)}(M) \to T_{\gamma(1)}(M)$ by $\tau^\gamma(A_0) = A(1)$,

Let $\tau\colon \gamma \mapsto \tau^\gamma$ with τ^γ being the parallel transport along γ defined by ∇ (see footnote $\|$ on page 6). The compatibility condition (3.1) is equivalent to the requirement that τ preserves the defined via g inner products along any path γ [43, 51], i.e.

$$g_{\gamma(0)} = g_{\gamma(1)} \circ (\tau^\gamma \times \tau^\gamma). \tag{3.2}$$

Since any parallel transport along paths (curves), like τ, is a (linear in our case) transport along paths [42], we see that equation (3.2), and hence (3.1), is equivalent to (2.6) for $s = 0,\ t = 1$ and $L = \tau$. Consequently, when analyzing the conditions for the validity of (3.1), we can apply proposition 2.2.

So, let g be a semi-pseudo-Riemannian metric on M, i.e. $g\colon x \mapsto g_x$ where $g_x\colon T_x(M) \times T_x(M) \to \mathbb{R},\ x \in M$ are symmetric bilinear real quadratic forms. Under what additional conditions on g there is a parallel transport τ such that the compatibility equation (3.2) holds?

Let $r(x)$ and $s(x)$ be respectively the rank and signature of $g(x)$ at x. Proposition 2.2, when applied to the bundle tangent to M, shows that a necessary and sufficient condition for the validity of (3.2) for some transport τ along paths is the independence of $r(x)$ and $s(x)$ of the point x at which they are calculated, i.e. $r(x) = \mathrm{const}$ and $s(x) = \mathrm{const}$ for every $x \in M$.

Therefore, if g has a constant rank and signature, there exists a linear transport τ along paths consistent with it. However, is this transport a parallel one, i.e does there exists a linear connection ∇ for which τ is a parallel transport (see footnote $\|$ on page 6)?

Let $\boldsymbol{F}^\gamma_s$ be the matrix corresponding to the map F^γ_s (see (2.4)) in some local bases. From [40, equation (4.9)] we know that the matrix of the coefficients of a linear transport L along paths, if it has a C^1 dependence on its parameters, is

$$\boldsymbol{\Gamma}(s;\gamma) := \left[\Gamma^i{}_j(s;\gamma)\right] = (\boldsymbol{F}^\gamma_s)^{-1}\,\frac{d\boldsymbol{F}^\gamma_s}{ds}. \tag{3.3}$$

The considerations in [40, section 5] show that L is the parallel transport corresponding to a linear connection with local coefficients $\Gamma^i{}_{jk}(x)$ iff

$$\boldsymbol{\Gamma}(s;\gamma) = \sum_{k=1}^{n} \boldsymbol{\Gamma}_k(\gamma(s))\dot\gamma^k(s) \tag{3.4}$$

where $\boldsymbol{\Gamma}_k(x) := \left[\Gamma^i{}_{jk}(x)\right]_{i,j=1}^{n}$, $n = \dim M$ and $\dot\gamma(s)$ is the vector tangent to γ at $\gamma(s)$.

Consequently, a necessary condition for τ^γ to be a parallel transport, assigned to some ∇, along γ is γ to be a C^1 path.

---

where $A_0 \in T_{\gamma(0)}(M)$ and the C^1 vector field A is given on $\gamma([0,1])$ via the initial-value problem $(\nabla_V A)|_{\gamma([0,1])} = 0$, $A(0) = A_0$. Here V is a vector field which on $\gamma([0,1])$ reduces to the vector field tangent to γ.

So, let γ be a C^1 path. Moreover, if $r(x) = n := \dim M$, i.e. if g is nondegenerate, we can reconstruct ∇ from g in the well known way [43, 50]. However, is it possible to be found a metric-compatible connection ∇ for a degenerate metric g, i.e. for $r(x) < n$? Surprisingly the answer to this question is positive. To prove this we need the following generalization of [41, proposition 2.5].

Proposition 3.1. *Let in the n-dimensional, $n < \infty$, real vector bundle (E, π, M) be given a fibre semi-pseudo-Riemannian metric g with constant rank and signature along every (resp. some fixed) path $\gamma \colon J \to M$. Let $\{\{e_i(\gamma(s)) : i = 1, \dots, n\}$ - basis in $E_{\gamma(s)}\}$ be a field of bases along γ in which g is represented by the matrix $\boldsymbol{G}(\gamma(s)) = [g_x(e_i(x), e_j(x))]_{x=\gamma(s)}$. Suppose $\boldsymbol{D}(\gamma(s))$ is a nondegenerate (real) matrix transforming $\boldsymbol{G}(\gamma(s))$ to a diagonal form by means of congruent transformation:*

$$\boldsymbol{D}^\top(\gamma(s))\boldsymbol{G}(\gamma(s))\boldsymbol{D}(\gamma(s)) = \boldsymbol{G}_0(\gamma(s)) := \operatorname{diag}(d_1(\gamma(s)), \dots, d_n(\gamma(s))) \tag{3.5}$$

where $\top$ means matrix transposition, p (=the number of positive eigenvalues of g) of the real numbers $d_1(\gamma(s)), \dots, d_n(\gamma(s))$ are positive, $q = r - p$(=the number of negative eigenvalues of g) of them are negative, and the remaining $n - r$(=the number of zero eigenvalues of g) of them are equal to zero. Then the set of all linear transports along paths compatible with g along every (resp. the given) path γ is described via the decomposition (2.4) *in which the matrix of the map F_s^γ has the form*

$$\boldsymbol{F}_s^\gamma = \boldsymbol{B}(\gamma)\boldsymbol{Z}(s;\gamma)(\boldsymbol{D}(\gamma(s)))^{-1} \tag{3.6}$$

for every (resp. the given) path γ. Here $\boldsymbol{B}(\gamma)$ is a nondegenerate $n \times n$ matrix function of γ and $\boldsymbol{Z}(s;\gamma)$ is any nondegenerate $n \times n$ matrix function of s and γ satisfying the equality

$$\boldsymbol{Z}^\top(s;\gamma)\boldsymbol{G}_0(s)\boldsymbol{Z}(s;\gamma) = \boldsymbol{G}_0(s). \tag{3.7}$$

Remark 3.1. By renumbering of the vectors of the local bases and renormalizing $\boldsymbol{D}$ and $\boldsymbol{Z}$ we can choose $\boldsymbol{G}_0$ in the form

$$\boldsymbol{G}_0(\gamma(s)) = \boldsymbol{G}_{p,q,n-r} := \operatorname{diag}(\underbrace{+1, \dots, +1}_{p \text{ times}}, \underbrace{-1, \dots, -1}_{q \text{ times}}, \underbrace{0, \dots, 0}_{(n-r) \text{ times}}).$$

In this case $\boldsymbol{Z}$ could be called a semi-pseudo-orthogonal matrix of type $(p, q, n - r)$ or of type (p, q) and defect $n - r$ (cf. the corresponding terminology concerning metrics [28]).

Remark 3.2. This proposition has an evident generalization for complex fibre bundles. In this case g is a semi-pseudo-Hermitian metric and the matrix transposition has to be replaced with Hermitian conjugation.

Proof. Mutatis mutandis this proof is an exact copy of the one of [41, proposition 2.5]. We have simply to make use of the new definition of $\boldsymbol{D}$ (via (3.5)) whose existence is proved, for instance, in [45, section 2.12]. □

Combining (3.3) with proposition 3.1, we conclude that a linear C^1 transport compatible with a metric g has coefficients whose matrix is of the form

$$\begin{aligned}\boldsymbol{\Gamma}(s;\gamma) &= \left\{\boldsymbol{Z}(s;\gamma)\boldsymbol{D}^{-1}(\gamma(s))\right\}^{-1}\frac{d}{ds}\left\{\boldsymbol{Z}(s;\gamma)\boldsymbol{D}^{-1}(\gamma(s))\right\}\\ &= \boldsymbol{D}(\gamma(s))\boldsymbol{Z}^{-1}(s;\gamma)\frac{d\boldsymbol{Z}(s;\gamma)}{ds}\boldsymbol{D}^{-1}(\gamma(s)) - \frac{d\boldsymbol{D}(\gamma(s))}{ds}\boldsymbol{D}^{-1}(\gamma(s)).\end{aligned} \tag{3.8}$$

Here have we supposed $\boldsymbol{Z}(s;\gamma)$ and $\boldsymbol{D}(x)$ to be of class C^1 with respect to s and x respectively, i.e. we have assumed that the metric g is of class C^1.

Now comparing (3.4) and (3.8), we see that a linear transport with coefficients given by (3.8) is a parallel transport for some linear connection ∇ iff

$$\boldsymbol{Z}(s;\gamma) = \widetilde{\boldsymbol{Z}}(\gamma(s)), \tag{3.9}$$

i.e. iff the matrix $\boldsymbol{Z}(s;\gamma)$ depends only on the point $\gamma(s)$ but not on s and γ separately. Besides, in this case ∇ has local coefficients given by

$$\boldsymbol{\Gamma}_k(x) = [\Gamma^i{}_{jk}(x)] = \boldsymbol{D}(x)\widetilde{\boldsymbol{Z}}^{-1}(x)\frac{\partial\widetilde{\boldsymbol{Z}}(x)}{\partial x^k}\boldsymbol{D}^{-1}(x) - \frac{\partial\boldsymbol{D}(x)}{\partial x^k}\boldsymbol{D}^{-1}(x) \tag{3.10}$$

where $\{x^k\}$ are local coordinates in a neighborhood of $x \in M$.

Notice, by (3.10) a necessary condition for L to be a parallel transport for a metric-compatible connection is the metric g to be of class C^1.

The overall above discussion can be summarized in the following theorem in which the above results are slightly generalize by introducing a set $U \subseteq M$.

Theorem 3.1. *Let on $U \subseteq M$ the semi-pseudo-Riemannian metric $g\colon x \mapsto g_x$ be defined by the symmetric C^1 quadratic forms $g_x\colon T_x(M)\times T_x(M) \to \mathbb{R}$, $x \in U$. A necessary and sufficient condition for the existence of a metric-compatible with g linear connection in U is the independence of the rank $r(x)$ and signature $s(x)$ of g_x of the point $x \in U$ at which they are calculated. Given g with these properties, the set of all such connections is selected via* (3.10)*. Moreover, the set of parallel transports corresponding to these connections coincides with the set of linear smooth (C^1) transports along smooth (C^1) paths which transports are compatible with g.*

4 When the space-time signature can change?

It is well known that general relativity is based on a pseudo-Riemannian metric of class C^2 over a 4-manifold V_4 [50] and the compatible with it torsionless linear connection, called Riemannian or Levi Civita's connection [50, 51], whose coefficients are the Christoffel symbols formed from the metric [43, 50]. By theorem 3.1 this metric must have a constant signature, conventionally assumed to be +2 or -2, or, in physical terms, $(-+++)$ or $(+---)$ respectively [50]. Thus, if one wants to build axiomatically general relativity, it is sufficient to suppose the existence of a torsionless metric-compatible connection or a constant signature of the pseudo-Riemannian metric.** Therefore in general relativity the signature is constant over the whole space-time.

In all known to the author (nonquantum) gravitational theories, the metrics, if any, are pseudo-Riemannian and of class C^1 or C^2 [49,50]. According to theorem 3.1, all such theories which use a metric-compatible connection, e.g. Einstein-Cartan and Einstein teleparallelism theories, must have a constant metric's signature. In other theories, such as Weyl's and metric-affine gravity, which are based on metric-incompatible connections, the signature is allowed, at least in principle, to change from point to point. Nevertheless that the last possibility potentially exists, it is not realized until now into a consistent gravitational theory that can stand experimental checking [49, 50, 53].

In the literature can be found papers devoted to the mathematical structure and possible physical events in space-time(s) with changing signature [4, 6, 7, 9, 10, 12, 15, 20, 21]. Most of them are based on modifications of general relativity [4–6, 8–10, 19, 24]. In such models the metric is globally only a symmetric quadratic form for which can exist sets on which it is degenerate or/and not differentiable or even discontinues. Excluding these peculiar sets, on the remaining parts (sets) of the space-time the metric is assumed, as usual, to be a symmetric, nondegenerate, and smooth (C^1 or C^2) quadratic form. On these latter sets, which can be called regular for the metric, is supposed to be valid general relativity. Consequently, by theorem 3.1, the signature of the metric is constant on them, but on the different sets it can be a different constant. On the sets on which the metric is degenerate the signature also can change from one to another set, depending on the number of zero eigenvalues of the metric on them. (Notice, the signature may have different values on the last sets and on the regular ones.) If on these sets, if any, the metric is still assumed to be symmetric, with a constant rank on them, and smooth (C^1 or C^2), then, due to theorem 3.1, on them can be suggested to be valid general relativity. Since in the last case

**In any one of these possibilities the concrete signature can be fixed via the equivalence principle [50, 52].

the three index Christoffel symbols (of second kind) do not exist, the connection coefficients have to be calculated by (3.10). At the end, there can be sets on which the smoothness of the metric breaks.[††] On them theorem 3.1 is not valid, metric-compatible connections do not exist, and, as a whole, on them a version of general relativity cannot be constructed. Hence these sets, if any, are the most probable ones on which the space-time signature may change.

5 Conclusion

In this work we have investigated the problem for possible change of the space-time signature from the view-point of existence of metric-compatible connections. Its main moral is: *on some space-time region there exists a metric-compatible connection if and only if the corresponding (degenerate or not) C^1 semi-pseudo-Riemannian metric has a constant signature and rank in this region.* For a globally defined metric with changing signature and, possibly, rank there is not a globally consistent with it linear connection, but on some subsets of the space-time such connection may exist.

When speaking about the space-time in this work, we implicitly suppose to be dealing with a classical, not quantum, theories of gravity in which it is a four-manifold. In quantum theories, like supergravity and string ones, the multidimensional, greater then four, character of space-time is accepted. In such theories there are possible (and necessary) transitions between geometries with changing space-time dimensions when the metric's signature and rank are variable [13, 54–56]. Since theories of this kind are based on mathematical structures different from linear connections, they do not fall in the subject of the present paper.

In connection with the metric-affine gravitational theories [49,53] the following question may arise. Given on the space-time (or on some its subset) an affine connection, is there a metric with which it is metric-compatible? This problem is, in some sense, opposite to the one investigated in the present work and will be considered elsewhere. Here we want only to mention that it can be solved completely by using the methods of this paper and [41, propositions 2.3 and 2.6] which can easily be generalized to describe degenerate metrics too.

Practically all of the mathematical results of this work, concerning the real case, can be generalized to the complex one. So, if required, they can be reformulated in terms of (complex or real) manifolds and (semi-)Hermitian (degenerate or not) metrics on them.

And a last remark. At a classical level, we know that there are three space and only one time dimension. This fact is reflected in the accepted,

[††]On these sets, usually, the metric is also degenerate as they often play a role of boundaries (sets of zero measure) between sets of the previous class [9, 10, 12, 13, 19, 24].

e.g. in general relativity, space-time signature. We also know from experience that under normal conditions, such as on the Earth or in the Solar system, the space-time signature is constant. This observation, combined with the results of the above investigation, leads to the conclusion that at present there are not experimental results which have to be described via space-time(s) with changing signature (and/or rank). In its turn, this conclusion makes the metric-compatible connections, maybe, the most effective mathematical tool for (nonquantum) description of gravitation.

Acknowledgments

The author expresses his gratitude to Nugzar V. Makhaldiani (Laboratory of Computing Technique and Automation, Joint Institute for Nuclear Research, Dubna, Russia) for paying his attention to the problem of signature change in 1994. He thanks Iskra S. Slavova (head of the OBK Chemical Laboratory, ED Balkanceramics Ltd., Novi Iskar, Bulgaria) without whose accidental help in April 1997 this work would not possibly be written.

This work was partially supported by the National Foundation for Scientific Research of Bulgaria under Grant No. F642.

References

[1] Hartle J. and Hawking S. Wave function of the Universe. *Phys. Rev.* ***D***, 28:2960, 1983.

[2] Sakharov A. Cosmological transitions with changes in the signature of the metric. *Sov. Phys. JETP*, 60:214, 1984.

[3] Wheeler J. A. On the nature of quantum geometrodynamics. *Ann. Phys.*, 2:604–614, 1957.

[4] Geroch R. P. Topology in general relativity. *J. Math. Phys.*, 8:782–786, 1967.

[5] Gibbons G. W. and Hartle J. B. Real tunneling geometries and the large-scale topology of the Universe. *Phys. Rev.* ***D***, 42(8):2458–2468, 1990.

[6] Gary T. Topology change in classical and quantum gravity. Preprint UCSBTH-90-51, 1990.

[7] Gibbons G. W. and Hawking S. W. Selection rules for topology change. *Comm. Math. Phys.*, 148:345–352, 1992.

[8] Hayward S. A. Signature change in general relativity. *Class. Quantum Grav.*, 9:1951–1962, 1992.

[9] Gibbons G. W. Topology and topology changing in general relativity. Preprint DAMTP/R-92/29, 1992.

[10] Alty L. J. Kleinian signature change. Preprint DAMTP R94/18, May 1984.

[11] Dray T., Manoque C., and Tucker R. Particle production from signature change. *General Relativity and Gravitation*, 23:967, 1993.

[12] Dray T., Manoque C. A., and Tucker R. W. The scalar field equation in the presence of signature change. *Phys. Rev.* ***D***, 48:2587–2590, 1993. (LANL xxx archive server, E-print No. gr-qc/9303002).

[13] Hayward S. A. Junction conditions for signature change. LANL xxx archive server, E-print No. gr-qc/9303034, 1993.

[14] Gibbons G. W. and Hawking S. W. Kinks and topology change. *Phys. Rev. Lett.*, 69:1719–1721, 1992.

[15] Alty L. J. The generalized Gauss-Bonnet-Chern theorem. Preprint DAMTP R94/39, 1994.

[16] Alty L.J. Binding blocks for topology change. Preprint DAMTP R94/38, 1994.

[17] Egusquiza I. L. Self-adjoint extensions and signature change. *Class. Quantum Grav.*, 12:L89–L92, 1995. (See also LANL xxx archive server, E-print No. gr-qc/9503015).

[18] Embacher F. Actions for signature change. *Phys. Rev.* ***D***, 51:6764–6777, 1995. (See also LANL xxx archive server, E-print No. gr-qc/9501004 and preprint UWThPh-1995-1).

[19] Alty L. J. Initial value problems and signature change. Preprint DAMTP R95/1, 1995.

[20] Alty L. J. and Fewster C. J. Initial value problems and signature change. *Class. Quant. Gravity*, 13:1129, 1996. (See also LANL xxx archive server, E-print No. gr-qc/9501026 and preprint DAMTP R95/1).

[21] Dray T., Ellis G., Helaby Ch., and Manogue C. Gravity and signature change. *General Relativity and Gravitation*, 29:591–597, 1997. (LANL xxx archive server, E-print No. gr-qc/9610063).

[22] Kriele M. Distinguished solutions for discontinuous signature change with weak junction conditions. LANL xxx archive server, E-print No. gr-qc/9610016, 1996.

[23] Hayward S. A. Signature change at material layers and step potentials. LANL xxx archive server, E-print No. gr-qc/9509052, 1995.

[24] Alty L. J. Kleinian signature change. *Class. Quantum Grav.*, 11:2523–2536, 1994.

[25] Dray T. Einstein equations in the presence of signature change. *J. Math. Phys.*, 37:5627, 1996. (LANL xxx archive server, E-print No. gr-qc/9610064).

[26] Greub W., Halperin S., and Vanstone R. *Connections, Curvature, and Cohomology*, volume 1. Academic Press, New York and London, 1972.

[27] Choquet-Bruhat Y. at el. *Analysis, manifolds and physics.* North-Holland Publ.Co., Amsterdam, 1982.

[28] Rosenfel'd B. A. *Non-Euclidean spaces.* Nauka, Moscow, 1969. (In Russian).

[29] *Mathematical encyclopedia*, volume 4. Vinogradov I. M., chief editor. Soviet encyclopedia, Moscow, 1984. (In Russian).

[30] Tricomi F. O. *On linear partial differential equations.* Gostekhizdat, Moscow, 1947. (Russian translation from Italian).

[31] Moĭseev E. I. *Equations of mixed type with spectral parameter.* Moscow University, Moscow, 1988. (In Russian).

[32] Friedrichs K. O. Symmetric positive linear differential equations. *Comm. Pure Appl. Math.*, 11(3):333–418, 1958.

[33] Friedrichs K. O. *Pseudo-differential operators. An introduction.* Lecture notes. Courant institute of mathematical sciences, New York, 1970.

[34] Smirnov M. M. *Degenerate elliptic and hyperbolic equations.* Nauka, Moscow, 1966. (In Russian).

[35] Oleĭnik O. A. and Rodkevich E. V. *Second order equations with non-negative characteristic form.* Review of science, Section Mathematics: Mathematical analysis 1969. VINITI, Moscow, 1971. (In Russian).

[36] Berezanskiĭ Yu. M. *Expansion of selfadjoint operators over eigenfunctions.* Naukova dumka, Kiev, 1965. (In Russian).

[37] Mikhlin S. G. *Linear partial differential equations.* Vȳshaya shkola, Moscow, 1977. (In Russian).

[38] Kuzmin' A. G. *Non-classical equations of mixed type and their application to gas dynamics.* Leningrad University, Leningrad, 1990. (In Russian).

[39] Gu Chaohao and Hong Jiaxing. *Some developments of the theory of partial differential equations of mixed type*, volume 90 of *Teubner Texte zur Mathematik*, pages 120–135. Teubner Verlag Geselschaft, Leipzig, 1986.

[40] Iliev B. Z. Linear transports along paths in vector bundles. I. General theory. JINR Communication E5-93-239, Dubna, 1993.

[41] Iliev B. Z. Linear transports along paths in vector bundles. IV. Consistency with bundle metrics. JINR Communication E5-94-17, Dubna, 1994.

[42] Iliev B. Z. Transports along paths in fibre bundles. II. Ties with the theory of connections and parallel transports. JINR Communication E5-94-16, Dubna, 1994.

[43] Kobayashi S. and Nomizu K. *Foundations of Differential Geometry*, volume I. Interscience Publishers, New York-London, 1963.

[44] Kobayashi S. and Nomizu K. *Foundations of Differential Geometry*, volume II. Interscience Publishers, New York-London-Sydney, 1969.

[45] Lankaster P. *Theory of matrices*. Academic Press, New York-London, 1969.

[46] Gantmacher F. R. *The theory of matrices*, volume one. Chelsea Pub. Co., New York, N.Y., 1960. (Translation from Russian. The right English transliteration of the author's name is Gantmakher).

[47] van der Waerden B. L. *Algebra II*. Springer-Verlag, Berlin-Heidelberg-New York, fifth edition, 1967. (In German).

[48] Gantmacher F. R. *The theory of matrices*, volume two. Chelsea Pub. Co., New York, N.Y., 1960 (reprinted 1964). (Translation from Russian. The right English transliteration of the author's name is Gantmakher).

[49] Gronwald F. and Hehl F. W. On gauge aspects of gravity. In Bergmann P. G., de Sabbata V., and Treder H.-J., editors, *Quantum Cosmology*, Proc. of the 14-th Course of the School of Cosmology and Gravitation, pages 148–198, Erice, Italy, May 1995, 1996. World Scientific, Singapore. (LANL xxx archive server, E-print No. gr-qc/9602013).

[50] Misner C. W., Thorne K. S., and Wheeler J. A. *Gravitation*. W. H. Freeman and Company, San Francisco, 1973.

[51] Bishop R. L. and Crittenden R. J. *Geometry of Manifolds*. Academic Press, New York-London, 1964.

[52] Iliev B. Z. Is the principle of equivalence a principle? *Journal of Geometry and Physics*, 24(3):209–222, 1997.

[53] Hehl F. W., McCrea J. D., Mielke E. W., and Ne'eman Y. Metric-affine gauge theory of gravity: field equations, Noether identities, world spinors, and breaking of dilation invariance. *Phys. Rep.*, 258(1 & 2):1–171, July 1995.

[54] Embacher F. Dimensionality, topology, energy, the cosmological constant, and signature change. *Class. Quantum Grav.*, 13:921, 1996. (See also LANL xxx archive server, E-print No. gr-qc/9504040 and preprint UWThPh-1995.11).

[55] Embacher F. Space-time dimension, Euclidean action and signature change. Preprint UWThPh-1995.28, 1995. (See also LANL xxx archive server, E-print No. gr-qc/9507041).

[56] Embacher F. Signature change induces compactification. *Phys. Rev.* ***D***, 52:2150, 1995. (See also LANL xxx archive server, E-print No. gr-qc/9410012 and preprint UWThPh-1994-47).